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Toward a NNLO calculation of the $\bar{B} \to X_s \gamma$ decay rate with a cut on photon energy: I. Two-loop result for the soft function

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Abstract

A theoretical analysis of the partial inclusive $\bar{B} \to X_s \gamma$ decay rate with a cut $E_\gamma \geqslant E_0$ on photon energy must deal with short-distance contributions associated with three different mass scales: the hard scale m_b, an intermediate scale $\sqrt{m_b \Delta}$, and a soft scale Δ, where $\Delta = m_b - 2E_0 \approx 1$ GeV for $E_0 \approx 1.8$ GeV. The cut-dependent effects are described in terms of two perturbative objects called the jet function and the soft function, which for a next-to-next-to-leading order analysis of the decay rate are required with two-loop accuracy. The two-loop calculation of the soft function is presented here, while that of the jet function will be described in a subsequent paper. As a by-product, we rederive the two-loop anomalous-dimension kernel of the B-meson shape-function.

1. Introduction

Weak-decay processes involving flavor-changing neutral currents are sensitive to the effects of new physics, because the decay amplitudes are loop-suppressed in the Standard Model. In this context, the decay $\bar{B} \to X_s \gamma$ plays an especially prominent role, since its rate is being measured increasingly well by the B-factories. The current experimental precision already matches the theoretical accuracy of the next-to-leading logarithmic prediction. This has triggered an effort to push the precision of the theoretical calculation of the decay rate in the Standard Model to the next level of accuracy. Due to the presence of several different scales in the decay process, this calculation involves a number of different elements.

Several of the steps required to achieve next-to-next-to-leading logarithmic (NNLO) accuracy have already been taken. The matching of the Standard Model onto an effective weak Hamiltonian has been completed by performing a three-loop matching calculation onto the electro- and chromo-magnetic dipole operators [1]. The effective weak Hamiltonian allows one to resum large perturbative logarithms of the form $\alpha_s \ln(M_W/m_b)$. To this end, the three-loop anomalous-dimension matrices for the four-quark operators [2] and for the mixing of the dipole operators among each other [3] have been calculated. The evaluation of the four-loop anomalous dimension of the mixing of the current–current operators into the dipole operators is in progress and is now the only missing element to obtain the Wilson coefficients in the effective weak Hamiltonian with NNLO precision. The most difficult part of the calculation is the evaluation of the matrix elements of the corresponding operators, in particular the ones involving penguin contractions of current–current operators with charm-quark loops. So far, the complete two-loop matrix element is known only for the electro-magnetic dipole operator $Q_{7\gamma}$ [4]. For other operators, only the parts proportional to $\beta_0 \alpha_s^2$ (more precisely, the terms proportional to $n_f \alpha_s^2$) have been obtained [5]. For the case of $Q_{7\gamma}$, not only the total decay rate but also the photon-energy spectrum

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has been evaluated at two-loop order [6]. For the remaining operators in the effective weak Hamiltonian, the photon spectrum is known to order $\beta_0\alpha_s^2$ [7].

An important motivation for undertaking a NNLO (i.e., order α_s^2) evaluation of the $\bar{B} \to X_s\gamma$ decay rate is the fact that such a calculation would reduce the strong dependence on the renormalization scheme adopted for the charm-quark mass, which is observed at NLO [8,9]. However, the charm mass is not the only low scale in the decay process. Another set of enhanced corrections arises because it is experimentally necessary to put a cut $E_\gamma > E_0$ on the photon energy (defined in the B-meson rest frame). The relevant scale is $\Delta = m_b - 2E_0$. For $E_0 \approx 1.8$ GeV, the currently lowest value of the cut achieved by the Belle experiment [10], the scale $\Delta \approx 1$ GeV is barely in the perturbative domain. For even higher values of E_0, the effects associated with the scale Δ cannot be calculated reliably in perturbation theory, in which case they are relegated into a non-perturbative shape-function [11,12].

As long as the cut energy is chosen sufficiently low, such that $m_b \gg \Delta \gg \Lambda_{\text{QCD}}$, the partial inclusive $\bar{B} \to X_s\gamma$ decay rate can be calculated perturbatively using a multi-scale operator-product expansion. It consists of a simultaneous expansion in powers of Δ/m_b and $\Lambda_{\text{QCD}}/\Delta$, combined with a systematic resummation of logarithms of ratios of the hard, intermediate, and soft scales [13] (see also [14]). At leading power in Δ/m_b and next-to-leading order in the expansion in powers of $\Lambda_{\text{QCD}}/\Delta$, it is possible to derive an exact expression for the partial decay rate $\Gamma(\Delta)$, valid to all orders in perturbation theory, in which the dependence on the variable Δ enters in a transparent way. The result is [15]

$$\Gamma(\Delta) = \frac{G_F^2\alpha}{32\pi^4}\left|V_{tb}V_{ts}^*\right|^2 m_b^3\bar{m}_b^2(\mu_h)\left|H_\gamma(\mu_h)\right|^2 U_1(\mu_h,\mu_i)U_2(\mu_i,\mu_0)\left(\frac{\Delta}{\mu_0}\right)^\eta \times\left\{\tilde{j}\left(\ln\frac{m_b\Delta}{\mu_i^2}+\partial_\eta,\mu_i\right)\tilde{s}\left(\ln\frac{\Delta}{\mu_0}+\partial_\eta,\mu_0\right)\frac{e^{-\gamma_E\eta}}{\Gamma(1+\eta)}\left[1-\frac{\eta(1-\eta)}{6}\frac{\mu_\pi^2}{\Delta^2}+\cdots\right]+\mathcal{O}\left(\frac{\Delta}{m_b}\right)\right\}. \tag{1}$$

Here m_b is the b-quark pole mass, and $\bar{m}_b(\mu)$ denotes the running mass defined in the $\overline{\text{MS}}$ scheme. The only hadronic parameter entering at this order is the quantity μ_π^2 related to the b-quark kinetic energy inside the B meson. The ellipses represent subleading corrections of order $(\Lambda_{\text{QCD}}/\Delta)^3$, which are unknown. The pole mass and μ_π^2 must be eliminated in terms of related parameters defined in a physical subtraction scheme, such as the shape-function scheme [16,17]. The scales $\mu_h \sim m_b$, $\mu_i \sim \sqrt{m_b\Delta}$, and $\mu_0 \sim \Delta$ are hard, intermediate, and soft matching scales. The hard function H_γ, the jet function $\tilde{j}$, and the soft function $\tilde{s}$ encode the contributions to the rate associated with these scales. Note that all information about the short-distance quantum fluctuations associated with the weak-interaction vertices in the effective weak Hamiltonian are contained in H_γ. Logarithms of ratios of the various scales are resummed into the evolution functions U_1 (evolution from the hard to the intermediate scale) and U_2 (evolution from the intermediate to the soft scale), as well as into the quantity

$$\eta = 2\int_{\mu_0}^{\mu_i}\frac{d\mu}{\mu}\Gamma_{\text{cusp}}\big[\alpha_s(\mu)\big], \tag{2}$$

which is given in terms of an integral over the universal cusp anomalous dimension of Wilson loops with light-like segments [18]. The result (1) is formally independent of the choices of the matching scales. In practice, a residual scale dependence remains because one is forced to truncate the perturbative expansions of the various objects in the formula for the decay rate. Reducing the scale uncertainty associated with the lowest short-distance scale, $\Delta \approx 1$ GeV, is the goal of the present work.

The soft function $\tilde{s}$ in (1) is related to the original B-meson shape-function $S(\omega,\mu)$ [11] through a series of steps. Starting from a perturbative calculation of the shape-function in the parton model with on-shell b-quark states, we first define

$$s\left(\ln\frac{\Omega}{\mu},\mu\right) \equiv \int_0^\Omega d\omega\, S_{\text{parton}}(\omega,\mu). \tag{3}$$

For $\Omega \gg \Lambda_{\text{QCD}}$, this parton-model expression gives the leading term in a systematic operator-product expansion of the integral over the true shape-function [15]. The first power correction is linked to the leading term by reparameterization invariance [19,20] and gives rise to the term proportional to μ_π^2/Δ^2 in (1). While the perturbative expression for S_{parton} involves singular distributions [16], the function s has a double-logarithmic expansion of the form

$$s(L,\mu) = 1+\sum_{n=1}^\infty\left(\frac{\alpha_s(\mu)}{4\pi}\right)^n\left(c_0^{(n)}+c_1^{(n)}L+\cdots+c_{2n-1}^{(n)}L^{2n-1}+c_{2n}^{(n)}L^{2n}\right). \tag{4}$$

The function $\tilde{s}$ is then obtained by the replacement rule [15]

$$\tilde{s}(L,\mu) \equiv s(L,\mu)\big|_{L^n\to I_n(L)}, \tag{5}$$

where $I_n(x)$ are nth order polynomials defined as

$$I_n(x)=\frac{d^n}{d\epsilon^n}\exp\left[\epsilon x+\sum_{k=2}^{\infty}\frac{(-1)^k}{k}\epsilon^k\zeta_k\right]\Bigg|_{\epsilon=0}. \tag{6}$$

By solving the renormalization-group equation for the soft function order by order in perturbation theory, the coefficients $c_{k\neq 0}^{(n)}$ of the logarithmic terms in (4) can be obtained from the expansion coefficients of the shape-function anomalous dimension and the β-function, together with the coefficients $c_0^{(n)}$ coefficients arising in lower orders [15]. The two-loop calculation performed in the present Letter gives the constant $c_0^{(2)}$ and provides a check on the two-loop anomalous dimension of the shape-function. We also note that from our result for $s(L,\mu)$ one can derive the two-loop expression for $S_{\text{parton}}(\omega,\mu)$ in terms of so-called star distributions [16].

In the next section, we discuss how to perform the two-loop calculation of the soft function s in an efficient way. The calculation is simplified by representing the δ-function operator appearing in the shape-function as the imaginary part of a light-cone propagator. In this way, we avoid having to deal with distribution-valued loop integrals and instead map the calculation to the evaluation of on-shell two-loop integrals with heavy-quark and light-cone propagators. Using integration-by-parts relations among these loop integrals, the entire calculation is reduced to the evaluation of four master integrals. After presenting the result for the bare soft function, we discuss its renormalization in Section 3. The relevant anomalous dimension depends both implicitly (through the coupling constant) and explicitly (through a star distribution) on the renormalization scale. This explicit dependence gives rise to Sudakov logarithms in the soft function. We conclude after presenting our final expression for the renormalized soft function in Section 4.

2. Two-loop calculation of the soft function

The definition of the soft function s in (3) implies that

$$s\left(\ln\frac{\Omega}{\mu},\mu\right)\equiv\int_0^{\Omega}d\omega\,\langle b_v|\bar h_v\delta(\omega+in\cdot D)h_v|b_v\rangle, \tag{7}$$

where h_v are effective heavy-quark fields in heavy-quark effective theory [21], b_v are on-shell b-quark states with velocity v, and n is a light-like 4-vector satisfying $n\cdot v=1$ (note that $v^2=1$ and $n^2=0$). The normalization of states is such that $\langle b_v|\bar h_v h_v|b_v\rangle=1$.

Working with the above representation of the soft function is difficult due to the presence of the δ-function differential operator, the Feynman rules for which involve δ-functions and their derivatives. This complication can be avoided by writing the δ-function operator as the discontinuity of a light-cone propagator in the background of the gluon field. This allows us to represent the soft function as a contour integral in the complex ω plane:

$$s\left(\ln\frac{\Omega}{\mu},\mu\right)=\frac{1}{2\pi i}\oint_{|\omega|=\Omega}d\omega\,\langle b_v|\bar h_v\frac{1}{\omega+in\cdot D+i0}h_v|b_v\rangle. \tag{8}$$

The Feynman rules for the gauge-covariant propagator involve light-cone propagators of the type $(\omega+n\cdot p)^{-1}$, which are straightforward to deal with using dimensional regularization and standard loop techniques. Dimensional analysis implies that an n-loop contribution to the matrix element in the integrand of the contour integral is proportional to $(-\omega)^{-1-2n\epsilon}$, where $d=4-2\epsilon$ is the dimension of space–time. The relevant contour integration yields

$$\frac{1}{2\pi i}\oint_{|\omega|=\Omega}d\omega\,(-\omega)^{-1-2n\epsilon}=-\Omega^{-2n\epsilon}\frac{\sin 2\pi n\epsilon}{2\pi n\epsilon}. \tag{9}$$

The two-loop Feynman diagrams contributing to the matrix element in (8) are shown in Fig. 1. They are on-shell heavy-quark self-energy diagrams with an operator insertion of the gauge-covariant light-cone propagator. Instead of drawing a separate diagram for each insertion, we draw the topology for a set of diagrams and indicate with a cross the locations where the operator can be inserted. The loop integrals arising in the calculation of the soft function contain heavy-quark as well as light-cone propagators. The one-loop master integral is

$$\int d^dk\,\frac{(-1)^{-a-b-c}}{(k^2+i0)^a(v\cdot k+i0)^b(n\cdot k+\omega+i0)^c}=i\pi^{\frac{d}{2}}2^b(-\omega)^{d-2a-b-c}I_1(a,b,c), \tag{10}$$

where $\omega\equiv\omega+i0$, and

$$I_1(a,b,c)=\frac{\Gamma(a+b-\frac{d}{2})\Gamma(2a+b+c-d)\Gamma(d-2a-b)}{\Gamma(a)\Gamma(b)\Gamma(c)}. \tag{11}$$

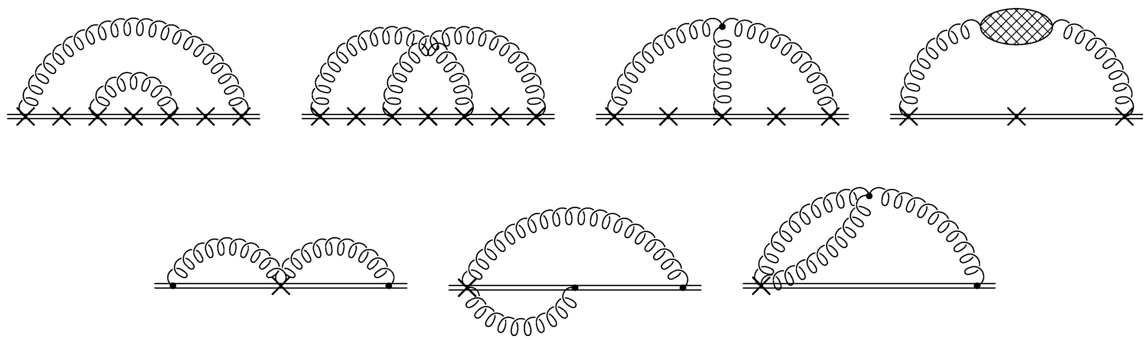

Fig. 1. Two-loop graphs contributing to the soft function. Double lines denote heavy-quark propagators, while crosses denote possible insertions of the operator $(\omega + in \cdot D + i0)^{-1}$.

The most general two-loop integral we need has the form

$$\int d^d k\, d^d l \frac{(-1)^{-a_1-a_2-a_3-b_1-b_2-b_3-c_1-c_2}}{(k^2)^{a_1}(l^2)^{a_2}[(k-l)^2]^{a_3}(v\cdot k)^{b_1}(v\cdot l)^{b_2}[v\cdot(k+l)]^{b_3}(n\cdot k+\omega)^{c_1}(n\cdot l+\omega)^{c_2}}$$
$$= -\pi^d 2^{b_1+b_2+b_3}(-\omega)^{2d-2a_1-2a_2-2a_3-b_1-b_2-b_3-c_1-c_2} I_2(a_1,a_2,a_3,b_1,b_2,b_3,c_1,c_2), \tag{12}$$

where all denominators have to be supplied with a "$+i0$" prescription. Note that we do not restrict the exponents $a_1, \dots, c_2$ to be positive. Loop integrals with non-trivial numerators are written as linear combinations of integrals for which some of the indices take negative values. A third light-cone propagator, $[n\cdot(k-l)+\omega]^{-1}$, can be eliminated using partial fractioning followed by a shift of the loop momenta. We use integration-by-parts identities [22] to reduce the two-loop integrals to a minimal set of master integrals. These linear algebraic identities are derived by observing that integrals over total derivatives vanish in dimensional regularization. With $I_2 \equiv I_2(a_1,a_2,a_3,b_1,b_2,b_3,c_1,c_2)$, the four relations obtained from applying each of the differential operators $\partial_{k_\mu} k^\mu$, $\partial_{k_\mu} l^\mu$, $\partial_{k_\mu} n^\mu$, and $\partial_{k_\mu} v^\mu$ to the integrand in (12) take the form

$$\begin{aligned}
0 &= \big[d - 2a_1 - a_3 - b_1 - c_1 - a_3\mathbf{a}_1^-\mathbf{a}_3^+ + a_3\mathbf{a}_2^-\mathbf{a}_3^+ - b_3\mathbf{b}_1^-\mathbf{b}_3^+ + c_1\mathbf{c}_1^+\big]I_2,\\
0 &= \big[a_3 - a_1 - a_1\mathbf{a}_1^+\mathbf{a}_2^- + a_1\mathbf{a}_1^+\mathbf{a}_3^- - a_3\mathbf{a}_1^-\mathbf{a}_3^+ + a_3\mathbf{a}_2^-\mathbf{a}_3^+ - b_1\mathbf{b}_1^+\mathbf{b}_2^- - b_3\mathbf{b}_2^-\mathbf{b}_3^+ + c_1\mathbf{c}_1^+ - c_1\mathbf{c}_1^+\mathbf{c}_2^-\big]I_2,\\
0 &= \big[a_1\mathbf{a}_1^+ - a_1\mathbf{a}_1^+\mathbf{c}_1^- - a_3\mathbf{a}_3^+\mathbf{c}_1^- + a_3\mathbf{a}_3^+\mathbf{c}_2^- + b_1\mathbf{b}_1^+ + b_3\mathbf{b}_3^+\big]I_2,\\
0 &= \big[a_1\mathbf{a}_1^+\mathbf{b}_1^- + a_3\mathbf{a}_3^+\mathbf{b}_1^- - a_3\mathbf{a}_3^+\mathbf{b}_2^- - 2b_1\mathbf{b}_1^+ - 2b_3\mathbf{b}_3^+ - c_1\mathbf{c}_1^+\big]I_2.
\end{aligned} \tag{13}$$

In these equations, the operator $\mathbf{a}_n^+$ ($\mathbf{a}_n^-$) raises (lowers) the index a_n by one unit. Four additional relations (with indices $1 \leftrightarrow 2$ interchanged) are obtained taking derivatives with respect to l_μ. Also, there are partial-fraction identities for integrals containing three different heavy-quark propagators, which follow from the relation $\mathbf{b}_1^- + \mathbf{b}_2^- - \mathbf{b}_3^- = 0$. Repeated application of these identities can be used to set at least one of the b_i exponents to zero.

The reduction to master integrals can be performed using computer algebra. To do so, one generates the equations for the index range relevant for a given calculation and uses Gaussian elimination to express complicated integrals in terms of simpler ones [23]. Since the number of equations is very large, the order in which the equations are solved is crucial, and [23] devised an efficient method to perform the reduction. A fast implementation of this algorithm is available in the form of a program that solves the relations (13) and generates a database containing the result for each integral [24]. At the end of the process, we are left with the master integrals

$$\begin{aligned}
M_1 &= I_2(1,0,1,0,1,0,0,1) = -\frac{2^{5-2d}\pi^3}{\sin^2(d\pi)\cos(d\pi)}\frac{1}{\Gamma^2(\frac{d-1}{2})},\\
M_2 &= I_2(1,0,1,0,1,0,1,0) = -\frac{\pi}{\sin(2d\pi)}\Gamma\left(4-\frac{3d}{2}\right)\Gamma\left(\frac{d}{2}-1\right),\\
M_3 &= I_2(1,0,1,1,1,0,0,1) = \frac{2^{2d-7}\pi^3}{\sin^2(d\pi)}\frac{\Gamma(7-2d)}{\Gamma^2(\frac{5-d}{2})},\\
M_4 &= I_2(1,1,0,1,1,0,1,1) = I_1(1,1,1)^2.
\end{aligned} \tag{14}$$

The first two are evaluated by using the well-known result for the one-loop self-energy integral to perform the first loop integration. The result takes the form (10) with non-integer exponents, so that the second loop integration is trivial. The only master integral that needs special consideration is the third one. Its Feynman parameterization is

$$M_3 = \Gamma(7-2d)\Gamma^2\left(\frac{d}{2}-1\right)\int_0^1 dx \int_0^1 dy\,(1-x)^{d-4}(1-y)^{3-d}y^{2d-7}(1-xy)^{1-\frac{d}{2}}. \tag{15}$$

We evaluate the parameter integral by expanding

$$(1-xy)^{1-\frac{d}{2}} = \sum_{n=0}^{\infty}(-1)^n \frac{\Gamma(2-\frac{d}{2})}{\Gamma(n+1)\Gamma(2-n-\frac{d}{2})}(xy)^n \tag{16}$$

and summing up the series after the integration.

With the integrals at hand, the evaluation of the diagrams is straightforward: first, each diagram is written in terms of integrals of the form (12), and then each integral is expressed in terms of the master integrals. Adding up the contributions of all graphs and expanding in $\epsilon = 2-\frac{d}{2}$, we find

$$\begin{aligned} s_{\text{bare}}(\Omega) &= 1 + \frac{Z_\alpha \alpha_s}{4\pi}\left(\frac{\Omega}{\mu}\right)^{-2\epsilon} C_F\left[-\frac{2}{\epsilon^2}+\frac{2}{\epsilon}-\frac{\pi^2}{6}+\left(\frac{\pi^2}{6}+\frac{2}{3}\zeta_3\right)\epsilon - \left(\frac{\pi^4}{80}+\frac{2}{3}\zeta_3\right)\epsilon^2 + \mathcal{O}(\epsilon^3)\right] \\ &\quad + \left(\frac{Z_\alpha \alpha_s}{4\pi}\right)^2\left(\frac{\Omega}{\mu}\right)^{-4\epsilon} C_F\left[C_F K_F(\epsilon) + C_A K_A(\epsilon) + T_F n_f K_f(\epsilon)\right] + \cdots, \end{aligned} \tag{17}$$

where

$$Z_\alpha = 1 - \beta_0 \frac{\alpha_s}{4\pi\epsilon} + \cdots \tag{18}$$

accounts for the renormalization of the bare coupling constant, and

$$\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f \tag{19}$$

is the first coefficient of the β-function. Throughout, $\alpha_s \equiv \alpha_s(\mu)$ is the renormalized coupling constant. Note that the bare soft function is scale independent, since the scale dependence of $\alpha_s(\mu)$ cancels against the explicit μ dependence. Also, since the soft function is an on-shell matrix element of a gauge-invariant operator it is gauge invariant. The terms of $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$ in the one-loop result above are needed for the evaluation of the counter-term contributions, as described in the next section. The two-loop coefficients are

$$\begin{aligned} K_F(\epsilon) &= \frac{2}{\epsilon^4} - \frac{4}{\epsilon^3} + \frac{2-\pi^2}{\epsilon^2} + \left(2\pi^2 - \frac{100}{3}\zeta_3\right)\frac{1}{\epsilon} - \pi^2 - \frac{59\pi^4}{60} + \frac{200}{3}\zeta_3 + \mathcal{O}(\epsilon), \\ K_A(\epsilon) &= -\frac{11}{6\epsilon^3} + \left(-\frac{1}{18}+\frac{\pi^2}{6}\right)\frac{1}{\epsilon^2} + \left(-\frac{55}{27} - \frac{23\pi^2}{36} + 9\zeta_3\right)\frac{1}{\epsilon} - \frac{326}{81} - \frac{361\pi^2}{108} + \frac{67\pi^4}{180} - \frac{85}{9}\zeta_3 + \mathcal{O}(\epsilon), \\ K_f(\epsilon) &= \frac{2}{3\epsilon^3} - \frac{2}{9\epsilon^2} + \left(-\frac{4}{27}+\frac{\pi^2}{9}\right)\frac{1}{\epsilon} - \frac{8}{81} - \frac{\pi^2}{27} - \frac{28}{9}\zeta_3 + \mathcal{O}(\epsilon). \end{aligned} \tag{20}$$

3. Renormalization of the soft function

As usual, we define an operator renormalization factor Z via

$$S(\omega,\mu) = \int d\omega'\, Z(\omega,\omega',\mu) S_{\text{bare}}(\omega'), \tag{21}$$

where Z absorbs the UV divergences of the bare soft function, such that the renormalized soft function is finite in the limit $\epsilon \to 0$. Here and below, all integrals over the variables ω, ω', etc. run from 0 to ∞. The convolution integral can be understood as a generalization of the matrix formula $O_i = Z_{ij} O_j^{\text{bare}}$, and so the usual relation between the Z factor and the anomalous dimension holds. It follows that

$$\gamma(\omega,\omega',\mu) = -\int d\omega'' \frac{dZ(\omega,\omega'',\mu)}{d\ln\mu} Z^{-1}(\omega'',\omega',\mu). \tag{22}$$

Below, we will write convolution integrals of this form using the short-hand notation $\gamma = -(dZ/d\ln\mu)\otimes Z^{-1}$. In the $\overline{\text{MS}}$ scheme, we have

$$Z(\omega,\omega',\mu) = \delta(\omega-\omega') + \sum_{k=1}^{\infty}\frac{1}{\epsilon^k} Z^{(k)}(\omega,\omega',\mu). \tag{23}$$

The Z factors depend on μ both implicitly via $\alpha_s(\mu)$, and explicitly via so-called star distributions related to the presence of Sudakov double logarithms. The latter dependence is a new feature, which leads to modifications of the standard relations derived, e.g., in [25]. Indeed, from the definition (22) of the anomalous dimension it follows that

$$\gamma + \gamma \otimes \sum_{k=1}^{\infty} \frac{Z^{(k)}}{\epsilon^k} = -\sum_{k=1}^{\infty} \frac{1}{\epsilon^k} \left[\frac{\partial Z^{(k)}}{\partial \alpha_s} \frac{d\alpha_s}{d \ln \mu} + \frac{\partial Z^{(k)}}{\partial \ln \mu} \right]. \tag{24}$$

Here $d\alpha_s/d\ln\mu = \beta(\alpha_s) - 2\epsilon\alpha_s$ is the generalized β-function in the regularized theory, and $\beta(\alpha_s)$ is the ordinary β-function. Both γ and β are independent of ϵ. Comparing coefficients of $1/\epsilon^k$, we then find

$$\gamma = 2\alpha_s \frac{\partial Z^{(1)}}{\partial \alpha_s}, \tag{25}$$

and

$$2\alpha_s \frac{\partial Z^{(n+1)}}{\partial \alpha_s} = 2\alpha_s \frac{\partial Z^{(1)}}{\partial \alpha_s} \otimes Z^{(n)} + \beta(\alpha_s) \frac{\partial Z^{(n)}}{\partial \alpha_s} + \frac{\partial Z^{(n)}}{\partial \ln \mu}, \quad n \geqslant 1. \tag{26}$$

While (25) is a familiar result [25], the relations (26) contain the additional $\partial Z^{(n)}/\partial \ln \mu$ piece, which is usually not present.

We now use the fact that, to all orders in perturbation theory, the anomalous dimension of the shape-function is given by [13,26]

$$\gamma(\omega, \omega', \mu) = -2\Gamma_{\rm cusp}(\alpha_s) \left(\frac{1}{\omega - \omega'} \right)_*^{[\mu]} + 2\gamma(\alpha_s)\delta(\omega - \omega'). \tag{27}$$

Here and below we encounter star distributions defined as [27]

$$\begin{aligned} \int_{\leqslant 0}^{\Omega} d\omega\, f(\omega) \left(\frac{1}{\omega} \right)_*^{[\mu]} &= \int_0^{\Omega} d\omega \frac{f(\omega) - f(0)}{\omega} + f(0) \ln \frac{\Omega}{\mu}, \\ \int_{\leqslant 0}^{\Omega} d\omega\, f(\omega) \left(\frac{\ln \frac{\omega}{\mu}}{\omega} \right)_*^{[\mu]} &= \int_0^{\Omega} d\omega \frac{f(\omega) - f(0)}{\omega} \ln \frac{\omega}{\mu} + \frac{f(0)}{2} \ln^2 \frac{\Omega}{\mu}, \end{aligned} \tag{28}$$

where $f(\omega)$ is a smooth test function. It follows from (25) that

$$Z^{(1)}(\omega, \omega', \mu) = -2Z_{\rm cusp}^{(1)}(\alpha_s) \left(\frac{1}{\omega - \omega'} \right)_*^{[\mu]} + 2Z_{\gamma}^{(1)}(\alpha_s)\delta(\omega - \omega'), \tag{29}$$

where

$$\Gamma_{\rm cusp}(\alpha_s) = 2\alpha_s \frac{\partial Z_{\rm cusp}^{(1)}}{\partial \alpha_s} = \sum_{n=0}^{\infty} \Gamma_n \left(\frac{\alpha_s}{4\pi} \right)^{n+1}, \qquad \gamma(\alpha_s) = 2\alpha_s \frac{\partial Z_{\gamma}^{(1)}}{\partial \alpha_s} = \sum_{n=0}^{\infty} \gamma_n \left(\frac{\alpha_s}{4\pi} \right)^{n+1}. \tag{30}$$

$\Gamma_{\rm cusp}$ is the cusp anomalous dimension already mentioned in connection with (2), whose two-loop expression has been derived in [18]. The other anomalous dimension, γ, has been calculated at two-loop order in [28,29]. The relevant expansion coefficients are

$$\begin{aligned} &\Gamma_0 = 4C_F, \qquad \Gamma_1 = C_F \left[\left(\frac{268}{9} - \frac{4\pi^2}{3} \right) C_A - \frac{80}{9} T_F n_f \right], \\ &\gamma_0 = -2C_F, \qquad \gamma_1 = C_F \left[\left(\frac{110}{27} + \frac{\pi^2}{18} - 18\zeta_3 \right) C_A + \left(\frac{8}{27} + \frac{2\pi^2}{9} \right) T_F n_f \right]. \end{aligned} \tag{31}$$

The relations (26) now allow us to express the coefficients $Z^{(k)}$ in terms of the expansion coefficients of β, $\Gamma_{\rm cusp}$, and γ. To derive these results, we need the following identities for star distributions (the first of which is somewhat laborious to derive)

$$\begin{aligned} &\left(\frac{1}{\omega - \omega''} \right)_*^{[\mu]} \otimes \left(\frac{1}{\omega'' - \omega'} \right)_*^{[\mu]} = 2 \left(\frac{\ln \frac{\omega - \omega'}{\mu}}{\omega - \omega'} \right)_*^{[\mu]} - \frac{\pi^2}{6} \delta(\omega - \omega'), \\ &\frac{d}{d \ln \mu} \left(\frac{1}{\omega - \omega'} \right)_*^{[\mu]} = -\delta(\omega - \omega'), \qquad \frac{d}{d \ln \mu} \left(\frac{\ln \frac{\omega - \omega'}{\mu}}{\omega - \omega'} \right)_*^{[\mu]} = -\left(\frac{1}{\omega - \omega'} \right)_*^{[\mu]}. \end{aligned} \tag{32}$$

Denoting by $Z_{[n]}$ the coefficient of $(\alpha_s/4\pi)^n$ in $Z(\omega, \omega', \mu)$, we obtain after some algebra

$$
\begin{aligned}
Z_{[0]} &= \delta(\omega-\omega'),\\
Z_{[1]} &= \delta(\omega-\omega')\left(\frac{\Gamma_0}{2\epsilon^2}+\frac{\gamma_0}{\epsilon}\right)-\frac{\Gamma_0}{\epsilon}\left(\frac{1}{\omega-\omega'}\right)_*^{[\mu]},\\
Z_{[2]} &= \delta(\omega-\omega')\left[\frac{\Gamma_0^2}{8\epsilon^4}+\frac{\Gamma_0(\gamma_0-\frac{3}{4}\beta_0)}{2\epsilon^3}+\left(\frac{\gamma_0(\gamma_0-\beta_0)}{2}+\frac{\Gamma_1}{8}-\frac{\pi^2}{12}\Gamma_0^2\right)\frac{1}{\epsilon^2}+\frac{\gamma_1}{2\epsilon}\right]\\
&\quad-\left(\frac{1}{\omega-\omega'}\right)_*^{[\mu]}\left[\frac{\Gamma_0^2}{2\epsilon^3}+\frac{\Gamma_0(\gamma_0-\frac{1}{2}\beta_0)}{\epsilon^2}+\frac{\Gamma_1}{2\epsilon}\right]+\frac{\Gamma_0^2}{\epsilon^2}\left(\frac{\ln\frac{\omega-\omega'}{\mu}}{\omega-\omega'}\right)_*^{[\mu]}.
\end{aligned}
\tag{33}
$$

This is the complete two-loop result for the renormalization factor of the B-meson shape-function.

According to (3), the soft function is defined as the integral over the renormalized (parton-model) shape-function. Using the fact that $Z(\omega,\omega',\mu)$ only depends on the difference $(\omega-\omega')$, we find that

$$
s\left(\frac{\Omega}{\mu},\mu\right)=\int_0^\Omega d\omega\, Z(\Omega,\omega,\mu)s_{\text{bare}}(\omega),
\tag{34}
$$

where $s_{\text{bare}}(\Omega)$ is defined as the integral over the bare shape-function and is scale independent. Expanding this relation in perturbation theory, we obtain

$$
\begin{aligned}
s_{[0]} &= s_{[0]}^{\text{bare}},\\
s_{[1]} &= Z_{[0]}\otimes s_{[1]}^{\text{bare}}+Z_{[1]}\otimes s_{[0]}^{\text{bare}},\\
s_{[2]} &= Z_{[0]}\otimes s_{[2]}^{\text{bare}}+Z_{[1]}\otimes s_{[1]}^{\text{bare}}+Z_{[2]}\otimes s_{[0]}^{\text{bare}},
\end{aligned}
\tag{35}
$$

with $s_{[0]}^{\text{bare}}=1$. The first term on the right-hand side in each line corresponds to the result obtained from the loop diagrams, given in (17). The remaining terms correspond to operator counter-terms. Explicitly, we obtain for the counter-term contributions

$$
\begin{aligned}
s_{[1]}^{\text{C.T.}} &= \frac{\Gamma_0}{2\epsilon^2}+\frac{\gamma_0}{\epsilon}-\frac{\Gamma_0}{\epsilon}\ln\frac{\Omega}{\mu},\\
s_{[2]}^{\text{C.T.}} &= \left[\frac{\Gamma_0}{2\epsilon^2}+\frac{\gamma_0}{\epsilon}-\frac{\Gamma_0}{\epsilon}\left(\ln\frac{\Omega}{\mu}-H_{-2\epsilon}\right)\right]s_{[1]}^{\text{bare}}(\Omega)+\frac{\Gamma_0^2}{8\epsilon^4}+\frac{\Gamma_0(\gamma_0-\frac{3}{4}\beta_0)}{2\epsilon^3}+\left(\frac{\gamma_0(\gamma_0-\beta_0)}{2}+\frac{\Gamma_1}{8}-\frac{\pi^2}{12}\Gamma_0^2\right)\frac{1}{\epsilon^2}+\frac{\gamma_1}{2\epsilon}\\
&\quad-\left[\frac{\Gamma_0^2}{2\epsilon^3}+\frac{\Gamma_0(\gamma_0-\frac{1}{2}\beta_0)}{\epsilon^2}+\frac{\Gamma_1}{2\epsilon}\right]\ln\frac{\Omega}{\mu}+\frac{\Gamma_0^2}{2\epsilon^2}\ln^2\frac{\Omega}{\mu},
\end{aligned}
\tag{36}
$$

where $H_{-2\epsilon}$ is the harmonic number, which results form the integral

$$
\int_0^1 dx\,\frac{1-x^{-2\epsilon}}{1-x}=H_{-2\epsilon}.
\tag{37}
$$

The counter-term contributions can be evaluated using the results for the bare one-loop soft function from (17) and the expressions for the anomalous-dimension coefficients given in (31). When adding these contributions to the result (17) for the bare soft function we find that all $1/\epsilon^n$ pole terms cancel, so that the limit $\epsilon\to 0$ can now be taken. In [15] the logarithmic terms in the renormalized soft function have been determined by solving the renormalization-group equation for the function s. At two-loop order, it was found that

$$
\begin{aligned}
s(L,\mu) &= 1+\frac{\alpha_s(\mu)}{4\pi}\left[c_0^{(1)}+2\gamma_0 L-\Gamma_0 L^2\right]+\left(\frac{\alpha_s(\mu)}{4\pi}\right)^2\left[c_0^{(2)}+\left(2c_0^{(1)}(\gamma_0-\beta_0)+2\gamma_1+\frac{2\pi^2}{3}\Gamma_0\gamma_0+4\zeta_3\Gamma_0^2\right)L\right.\\
&\quad\left.+\left(2\gamma_0(\gamma_0-\beta_0)-c_0^{(1)}\Gamma_0-\Gamma_1-\frac{\pi^2}{3}\Gamma_0^2\right)L^2+\left(\frac{2}{3}\beta_0-2\gamma_0\right)\Gamma_0 L^3+\frac{\Gamma_0^2}{2}L^4\right],
\end{aligned}
\tag{38}
$$

where the non-logarithmic one-loop coefficient reads [16,30]

$$
c_0^{(1)}=-\frac{\pi^2}{6}C_F.
\tag{39}
$$

Our results for the logarithmic terms agree with (38) and thus confirm the existing results for the two-loop anomalous dimensions Γ_1 [18] and γ_1 [28,29]. In addition, our calculation gives for the non-logarithmic piece at two-loop order the expression

$$
c_0^{(2)}=C_F^2\left(-\frac{4\pi^2}{3}-\frac{3\pi^4}{40}+32\zeta_3\right)+C_F C_A\left(-\frac{326}{81}-\frac{427\pi^2}{108}+\frac{67\pi^4}{180}-\frac{107}{9}\zeta_3\right)+C_F T_F n_f\left(-\frac{8}{81}+\frac{5\pi^2}{27}-\frac{20}{9}\zeta_3\right).
\tag{40}
$$

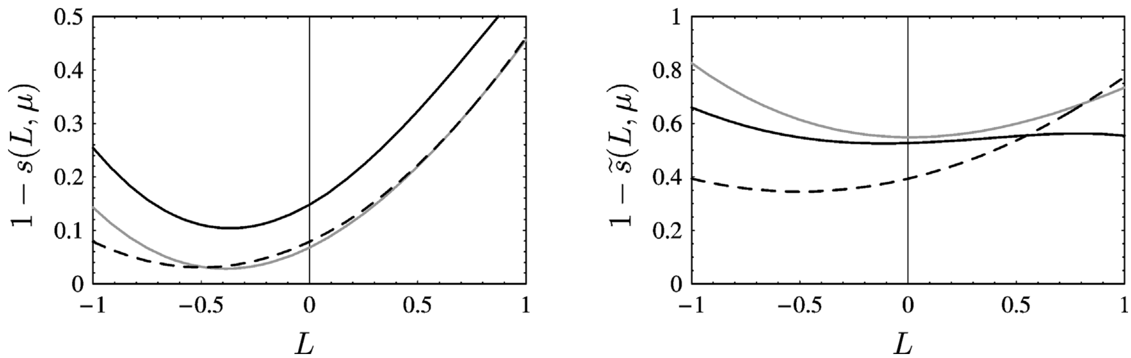

Fig. 2. One- and two-loop corrections to the soft functions $s(L, \mu)$ and $\tilde{s}(L, \mu)$ evaluated at $\alpha_s(\mu) = 0.45$. The dashed lines show the one-loop results, while the solid lines give the complete two-loop results derived in the present work. The gray lines are obtained if only the $\beta_0 \alpha_s^2$ terms are kept in the two-loop contributions.

This is the main result of the present work.

It is interesting to compare the exact answer for the coefficient $c_0^{(2)}$ with the approximation obtained by keeping only the terms of order $\beta_0 \alpha_s^2$, which are much simpler to derive than the full result given in the present Letter. In the absence of exact two-loop results, it is sometimes argued that the $\beta_0 \alpha_s^2$ terms constitute the dominant part of the complete two-loop correction. In the present case, we obtain for $N_c = 3$ colors (note that $\beta_0 = 9$ for $n_f = 3$ light flavors)

$$c_0^{(2)} \approx 8.481 \cdot \frac{\beta_0}{9} - 62.682 \approx -54.201. \tag{41}$$

Keeping only the $\beta_0 \alpha_s^2$ term would give 8.481, which has the wrong sign and is off by almost an order of magnitude. This illustrates the importance of performing exact two-loop calculations.

4. Discussion and summary

Having completed the two-loop calculation of the soft function, we now briefly discuss the impact of our results for the prediction of the partial inclusive $\bar{B} \to X_s \gamma$ branching ratio. We begin by displaying the final expressions for the functions $s(L, \mu)$ and $\tilde{s}(L, \mu)$ obtained from (38) and (5) for the case of $N_c = 3$ colors and $n_f = 3$ light quark flavors. We obtain

$$\begin{aligned} s(L, \mu) \approx {} & 1 + \left(-0.175 - 0.424L - 0.424L^2\right)\alpha_s(\mu) \\ & + \left(-0.343 - 0.201L - 0.433L^2 + 0.383L^3 + 0.090L^4\right)\alpha_s^2(\mu) + \cdots, \\ \tilde{s}(L, \mu) \approx {} & 1 + \left(-0.873 - 0.424L - 0.424L^2\right)\alpha_s(\mu) \\ & + \left(-0.660 + 0.821L + 0.456L^2 + 0.383L^3 + 0.090L^4\right)\alpha_s^2(\mu) + \cdots. \end{aligned} \tag{42}$$

The two-loop corrections are quite significant, especially in the case of $\tilde{s}(L, \mu)$. Fig. 2 shows the dependence of the soft functions on $L = \ln(\Omega/\mu)$ at the fixed renormalization scale μ chosen such that $\alpha_s(\mu) = 0.45$, corresponding to a renormalization point $\mu \approx 1.1$ GeV as appropriate for the photon-energy cuts used in current measurements of the $\bar{B} \to X_s \gamma$ decay rate. In addition to the one- and two-loop results, we display the results obtained if only terms of order $\beta_0 \alpha_s^2$ are kept in the two-loop coefficients. The figure shows that the two-loop effects calculated in this Letter can have an impact on the soft functions at the 10–20% level, and that keeping only the $\beta_0 \alpha_s^2$ terms does, in general, not provide an accurate description of the two-loop effects. We stress, however, that the large perturbative corrections seen in the figure do not translate in similarly large corrections to the $\bar{B} \to X_s \gamma$ decay rate. The size of the corrections is strongly reduced once the pole mass in (1) is eliminated in favor of a low-scale subtracted b-quark mass, such as the shape-function mass [16,17]. At one-loop order this was demonstrated in [13], and we expect similar cancelations to persist in higher orders. Note also that the soft functions by themselves are not renormalization-group invariant, so it is meaningless to study their dependence on the scale μ for fixed Ω. In physical results such as the expression for the $\bar{B} \to X_s \gamma$ decay rate in (1), the scale dependence of the soft function cancels against the μ_0 dependence of the objects U_2, $(\Delta/\mu_0)^\eta$, and η in (2). A detailed analysis of the phenomenological impact of NNLO corrections on the $\bar{B} \to X_s \gamma$ decay rate will be given elsewhere.

In summary, we have calculated the two-loop expression for the soft function $s(L, \mu)$, which is defined in terms of an integral over the B-meson shape-function in the parton model. This quantity is a necessary ingredient for the NNLO evaluation of the $\bar{B} \to X_s \gamma$ decay rate with a cut on the photon energy. For a sufficiently low cut energy, the partial inclusive decay rate can be calculated in a multi-scale operator-product expansion, in which the soft function arises in the final expansion step, when a current–current correlator in soft-collinear effective theory is matched onto bilocal heavy-quark operators in heavy-quark effective theory. If the cut on the photon energy is so severe that the contribution from the soft region cannot be evaluated perturbatively, the soft part should be subtracted from the partonic result for the cut rate, before it is convoluted with the renormalized shape-function. Even in that case our result is a necessary component in the consistent calculation of the $\bar{B} \to X_s \gamma$ photon spectrum at NNLO. Moreover, since the soft function is universal to all inclusive heavy-to-light decays in the end-point region, our results are also relevant for semi-leptonic $\bar{B} \to X_u l^- \bar{\nu}$ decay spectra, if one attempts to extend the analysis of [16,31] to NNLO.

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